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WAVES AGAINST A CLIFF OVERHANGING AT AN ANGLE OF 135°

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WAVES AGAINST A CLIFF OVERHANGING AT AN ANGLE OF 135°

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We apply the method of Lewy* to treat numerically waves progressing against a cliff overhanging at 135°. We then analyze the nature of the waves by computing the shape of the two fundamental standing waves, φ_0 and φ_1, from which all progressing wave solutions can be constructed. The principle results are the following:** at shore the amplitude damping effect of the cliff is about 10 percent, while the wave length is 3 percent shorter. Beyond one wave length from the shore the effect of the cliff is negligible (see Figure 4).

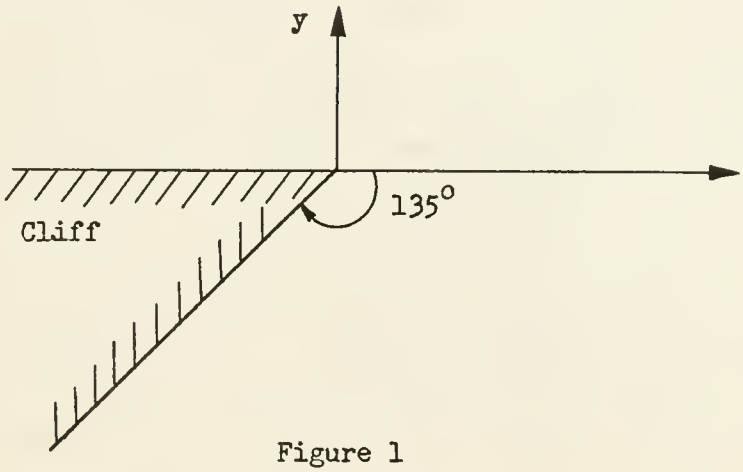

Figure 1

We seek two potential functions φ_0 and φ_1 which have the following properties:

(A) They satisfy the boundary conditions

1) $\varphi_y - \varphi = 0$ on $y = 0$, $x > 0$

2) $\varphi_\eta = 0$ on the cliff $y = x$, $y < 0$ ($\frac{\partial}{\partial \eta}$ = normal derivative)

(B) In addition, φ_0 and φ_1 should behave at ∞ like $\mathrm{Re}(e^{i\alpha} e^{-iz})$ and $\mathrm{Re}(e^{i(\alpha + \frac{\pi}{2})} e^{-iz})$ so that we can obtain a progressing wave, Φ, moving in from infinity by setting $\Phi = \varphi_0 \sin t + \varphi_1 \cos t$.

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* The general solution for angles $\frac{p\pi}{2n}$ is given in H. Lewy - "Water Waves on Sloping Beaches." Bulletin of the American Mathematical Society, September 1946. Our case corresponds to $p = 3$, $n = 2$.

** For a summary of the basic underlying theory, together with a brief discussion of the contents of the present and a number of other reports on problems in the same general field, see IMM-NYU 160 - "Waves over Beaches of Small Slope, Under a Dock, Under an Overhanging Cliff, and Past Plane Barriers."

(C) We permit φ_1 to have a logarithmic singularity at the origin, but we require that φ_o be finite at the origin.

The boundary conditions may be formulated for the analytic functions $\chi_o(z)$ and $\chi_1(z)$ which correspond to φ_o and φ_1 (i.e., φ is the real part of χ):

3) $$\mathrm{Im}(D + i)\chi = 0 \quad \text{for} \quad z = x,\ x > 0,\ \text{with } D = \frac{d}{dz};$$

4) $$\mathrm{Im}(e^{-i\pi 3/4}D)\chi = 0 \quad \text{for} \quad z = re^{i\pi 3/4},\ r > 0.$$

The functions φ_o and φ_1 are obtained from suitable combinations of solutions of the following linear ordinary differential equations:

5) $$(D - 1)(D + i)f_o(z) = 0, \qquad z = x + iy$$

6) $$(D - 1)(D + i)f_1(z) = \frac{A}{z^{2/3}}, \qquad A = \text{real constant}$$

7) $$(D - 1)(D + i)f_{-1}(z) = \frac{B}{z^2}, \qquad B = \text{real constant.}$$

The right hand sides have the property that they are real on the surface and on the reflection of the surface in the cliff, while the left hand sides are the operators which correspond to the boundary conditions on the surface and the reflection of the surface in the cliff.

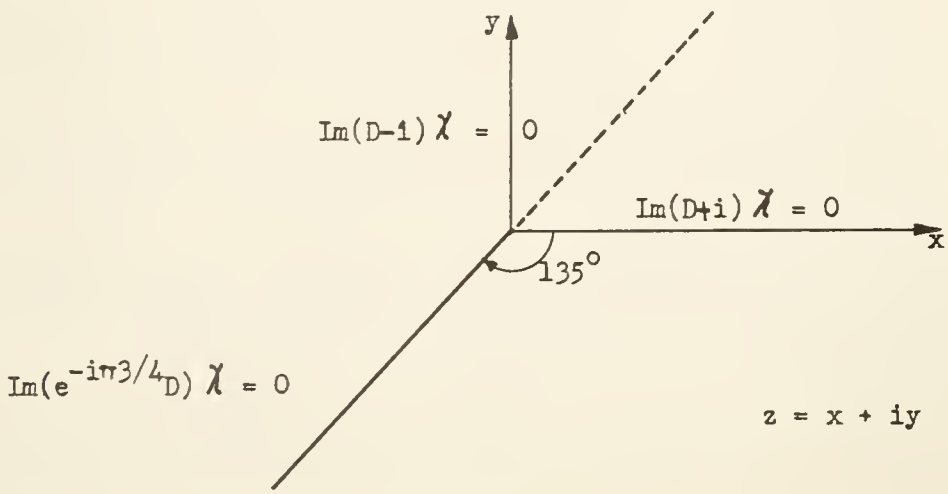

Figure 2

The solution f_o, f_1, f_{-1} which satisfy the boundary conditions* 3) and 4) are:

8) $$f_o = e^{i\pi 3/4}\, e^{z} + e^{-i\pi 3/4}\, e^{-iz}.$$

9) $$f_1 = e^{i\pi 3/4}\, e^{z} \int_o^{z} e^{-t} t^{\frac{1}{3}}\, dt + e^{-i\pi 3/4}\, e^{-iz} \int_o^{-iz} e^{-t} t^{\frac{1}{3}}\, dt$$

10) $$f_{-1} = e^{i\pi 3/4} e^{z} \int_1^{z} \frac{e^{-t}}{t}\, dt + e^{-i\pi 3/4}\, e^{-iz} \int_1^{-iz} \frac{e^{-t}}{t}\, dt\ ,$$

where the integrals $\int_o^{-iz} e^{-t}\, t^{\frac{1}{3}}\, dt$ and $\int_1^{-iz} \frac{e^{-t}}{t}\, dt$ are taken over a path which keeps the origin to the left (or $-iz = e^{i\pi 3/2} z$).

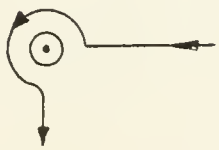

Figure 3

After studying the asymptotic behavior** of the functions f_j, we set

11) $$\chi_o = \frac{1}{\sqrt{3}} \left[f_o - \frac{f_1}{\Gamma(\frac{4}{3})} \right]$$

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* See Appendix - Part A

** See Appendix - Part B

12) $$\chi_1 = \frac{1}{\pi}\left[f_{-1} - f_o \int_1^{\infty} \frac{e^{-t}}{t}\, dt\right] + \chi_o$$

and find that

13) $$\chi_o \cong e^{-i\pi\, 11/12}\, e^{-iz}\ , \quad \chi_1 \cong e^{-i\pi\, 5/12}\, e^{-iz} \quad \text{as } |z| \to \infty .$$

We then obtain φ_o and φ_1 as the real parts of the functions χ_o and χ_1. The graphs of the values of φ_o and φ_1 on $y = 0$ are shown in Figure 4. The conclusions of the first paragraph follow from this figure.

The asymptotic formulas for φ_o and φ_1 obtained from (13) give results differing from the actual values of φ_o and φ_1 by less than one percent for $x \geq 5$.

A short table of the integrals appearing in f_1 together with a description of the method of computation is in Part C of the appendix.

We observe that the ratio

$$\frac{\text{amplitude of } \varphi_o \text{ at origin}}{\text{amplitude of } \varphi_o \text{ at infinity}} = \sqrt{\frac{2}{3}} = \sqrt{\frac{n}{p}}.\ ^{*}$$

The damping effect of an overhanging cliff is explained by the fact that this ratio is less than 1.

On the other hand, the first maximum of φ_o (and φ_1) is at least one percent higher than the maximum amplitude at ∞. Further study of this problem is necessary, probably along the lines of the work of K. O. Friedrichs, in order to obtain an overall picture of the path of the crests of the progressing wave, Φ.

The progressing wave is of the form

$$\Phi = \varphi_o \sin t + \varphi_1 \cos t = e^y \cos\left(x + \frac{5}{12}\pi + t\right), \text{ for large } x.$$

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*The general result, $\sqrt{\frac{n}{p}}$ was obtained by H. Lewy in his paper previously cited.

Standing Wave Solution on Surface.

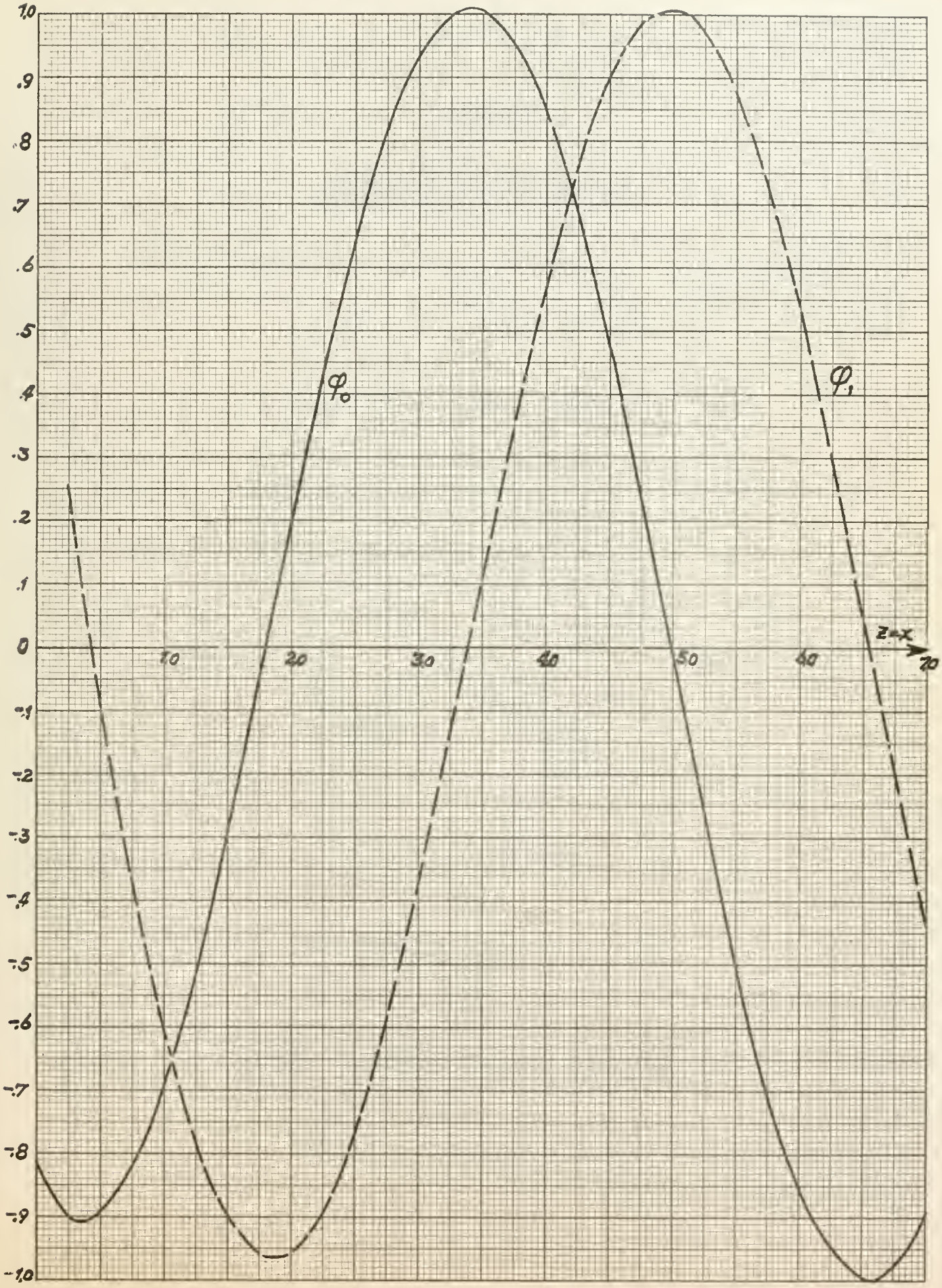

Figure 4.

APPENDIX

Part A - Boundary Conditions

We now verify that the functions f_j satisfy the boundary conditions

3) $$\mathrm{Im}(D + i)f_j = 0 \qquad \text{for } z = x,\ x > 0$$

4) $$\mathrm{Im}(e^{-i\pi 3/4})f_j = 0 \qquad \text{for } z = re^{-i\pi 3/4},\ r > 0.$$

3_o) $(D + i)f_o = (1 + i)e^{i\pi 3/4}e^z = \sqrt{2}\,e^z$ which is real for $z = x > 0$.

4_o) $(e^{-i\pi 3/4}D)f_o = e^z + e^{-i\pi}\,e^{-iz}$, which for $z = re^{-i\pi 3/4}$, becomes

$$= e^{re^{i\pi 1/4}} + e^{re^{-i\pi 1/4}}, \text{ which is real.}$$

3_1) $$(D + i)f_1 = (1 + i)e^{i\pi 3/4}\,e^z \int_0^z e^{-t}t^{1/3}dt + e^{i\pi 3/4}\,z^{1/3} + e^{i\pi 3/4}(-iz)^{1/3}$$

$$= \sqrt{2}\,e^z \int_0^z e^{-t}t^{1/3}dt + (e^{i\pi 3/4}\,z^{1/3} + e^{-i\pi 3/4}\,z^{1/3})$$

which is real. It is here that we determine the path of integration so that $-iz = e^{i\pi 3/2}z$.

$$4_1)\quad (e^{-i\pi 3/4}D)f_1 = e^z \int_0^z e^{-t}\, t^{1/3}\, dt + e^{-iz} \int_0^{-iz} e^{-t}\, t^{1/3} + (-iz)^{1/3} + z^{1/3};$$

where $z = re^{-i\pi 3/4}$

$$= e^{re^{-i\pi 3/4}} \int_0^{re^{-i\pi 3/4}} e^{-t}\, t^{1/3}\, dt$$

$$+ e^{re^{i\pi 3/4}} \int_0^{re^{i\pi 3/4}} e^{-t}\, t^{1/3}\, dt$$

$$+ e^{-i\pi 1/4}\, r^{1/3} + e^{i\pi 1/4}\, r^{1/3}$$

which is real. (We have again made use of the definition $-iz = e^{i\pi 3/2}z$.)

$$3_{-1})\quad (D + i)f_{-1} = (1 + i)e^{i\pi 3/4}\, e^z \int_1^z \frac{e^{-t}}{t}\, dt + \left(\frac{e^{i\pi 3/4}}{z} + \frac{e^{-i\pi 3/4}}{z}\right)$$

which is real for z real.

$$4_{-1})\ (e^{-i\pi 3/4}D)f_{-1} = e^z \int_1^z \frac{e^{-t}}{t}\, dt + e^{-iz} \int_1^{-iz} \frac{e^{-t}}{t}\, dt + \frac{1}{z} + \frac{i}{z}\ ;\ z = re^{-i\pi 3/4}$$

$$= e^{re^{-i\pi 3/4}} \int_1^{re^{-i\pi 3/4}} \frac{e^{-t}}{t}\, dt$$

$$+ e^{re^{i\pi 3/4}} \int_1^{re^{i\pi 3/4}} \frac{e^{-t}}{t}\, dt$$

$$+ \frac{e^{-i\pi 3/4}}{r} - \frac{e^{i\pi 3/4}}{r},$$ which is real.

Part B - Behavior at Infinity

We cannot use the functions f_j individually to obtain standing waves, because these functions are unbounded at infinity. However, we study the nature of the f_j at infinity in order to decide whether we can choose suitable linear combinations of them as the functions χ_o and χ_1. We need only consider the integrals appearing in f_1 and f_{-1}.

We begin with

$$\int_o^z e^{-t}\, t^{1/3}\, dt = \int_o^\infty e^{-t}\, t^{1/3}\, dt - \int_z^\infty e^{-t}\, t^{1/3}\, dt,$$

so that by integration by parts of the second term, we obtain

$$14) \qquad \int_o^z e^{-t}\, t^{1/3}\, dt = \Gamma(\tfrac{4}{3}) - e^{-z}\left[z^{1/3} + \frac{a}{z^{2/3}} + \dots \right] .$$

On the other hand,

$$\int_o^{-iz} e^{-t}\, t^{1/3}\, dt = \int_o^\infty e^{-t}\, t^{1/3}\, dt - \int_{-iz}^{\infty -iz} e^{-t}\, t^{1/3}\, dt,$$

where in the first integral on the right hand side of the equation $t = e^{2\pi i} t$ and in the second integral $-iz = e^{i\pi 3/2} z$.

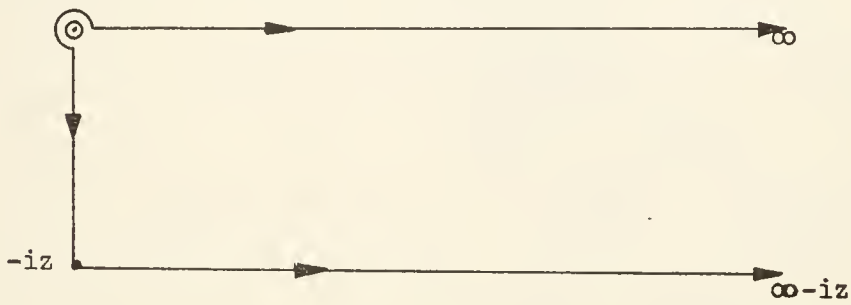

Figure 5

Consequently, integration by parts of the second term yields

$$15)\qquad \int_0^{-iz} e^{-t}\, t^{\frac{1}{3}} dt \cong e^{\frac{2\pi i}{3}} \Gamma\left(\tfrac{4}{3}\right) - e^{iz}\left[(-iz)^{\frac{1}{3}} + \frac{b}{(-iz)^{2/3}} + \dots \right]$$

From 14), 15) and 11) we find that $\chi_0 \cong e^{\frac{-i\pi 11}{12}} e^{-iz}$, which verifies the statement of equation 13).

Next we study the integrals in f_{-1}:

$$\int_1^z \frac{e^{-t}}{t}\, dt = \int_1^\infty \frac{e^{-t}}{t}\, dt - \int_z^\infty \frac{e^{-t}}{t}\, dt$$

and integration by parts of the second integral gives us

$$16)\qquad \int_1^z \frac{e^{-t}}{t}\, dt \cong \int_1^\infty \frac{e^{-t}}{t}\, dt - e^{-z}\left[\frac{1}{z} + \frac{c}{z^2} + \dots \right].$$

The remaining integral can be handled as follows:

$$\int_1^{-iz} \frac{e^{-t}}{t}\, dt = \int_1^\infty \frac{e^{-t}}{t}\, dt - \int_{-iz}^\infty \frac{e^{-t}}{t}\, dt + 2\pi i.$$

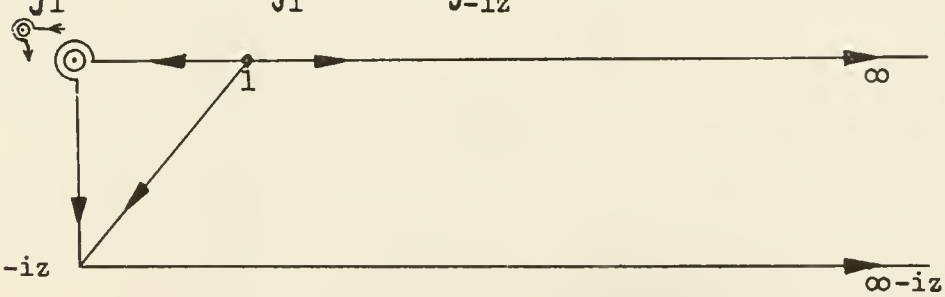

Figure 6

The number $2\pi i$ is added because the integrand has a simple pole at the origin which is circled in the positive direction (see Fig. 6). We can now write down the expansion

$$17)\quad \int_1^{-iz} \frac{e^{-t}}{t}\, dt \cong \int_1^{\infty} \frac{e^{-t}}{t}\, dt + 2\pi i - e^{iz}\left[\frac{1}{-iz} + \frac{d}{(-iz)^2} + \cdots\right].$$

With the results of 14), 15), 16) and 17) it is a simple matter to verify that the definition of χ_1 in 12) leads to

$$\chi_1 \cong e^{-i\pi 5/12}\, e^{-iz}.$$

Part C - Table of Exponential Integrals

The integrals appearing in f_1 were evaluated as follows:

(a) for $|z| \leq 1$, we replaced e^{-t} by its power series about the origin, multiplied by $t^{1/3}$ and then integrated term by term.

(b) for $|z| \geq 1$, we evaluated $e^{-t}t^{1/3}$, $\sin t \cdot t^{1/3}$, and $\cos t \cdot t^{1/3}$ at intervals of $\Delta t = .25$ and used numerical integration up to $t = 7$.

A short table of values of these integrals correct to within 1 unit in the fourth decimal place may prove useful in similar computations.

z	$\int_0^{ze^{i\pi 3/2}} e^{-t}t^{1/3}dt$ Real	Imaginary	$\int_0^{z} e^{-t}t^{1/3}dt$
0.00	0.0000	0.0000	0.0000
.25	+.1166	+.0168	.1026
.50	+.2829	+.0831	.2258
.75	+.4552	+.2078	.3400
1.00	+.6076	+.3914	.4396
1.25	+.7190	+.6255	.5241
1.50	+.7726	+.8974	.5944
1.75	+.7564	+1.1901	.6524
2.00	+.6641	+1.4833	.6997
2.25	+.4952	+1.7557	.7382
2.50	+.2554	+1.9863	.7693
2.75	-.0438	+2.1560	.7943
3.00	-.3858	+2.2492	.8144
3.25	-.7503	+2.2550	.8305
3.50	-1.1142	+2.1683	.8434
3.75	-1.4532	+1.9901	.8536
4.00	-1.7440	+1.7278	.8618
4.25	-1.9655	+1.3949	.8683
4.50	-2.1003	+1.0100	.8734
4.75	-2.1364	+ .5961	.8775
5.00	-2.0678	+ .1789	.8808
5.25	-1.8953	- .2148	.8833
5.50	-1.6265	- .5591	.8854
5.75	-1.2756	- .8302	.8870
6.00	- .8626	-1.0087	.8882
6.25	- .4121	-1.0804	.8892
6.50	+ .0483	-1.0379	.8900
6.75	+ .4895	- .8808	.8907
7.00	+ .8829	- .6162	.8912

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